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# Phase Space Localization and Approach to Thermal Equilibrium for a Class of Open Systems

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**Abstract**

We analyse the evolution of a quantum oscillator in a finite temperature environment using the quantum state diffusion (QSD) picture. Following a treatment similar to that of reference [7] we identify stationary solutions of the corresponding Itô equation. We prove their global stability and compute typical time scales characterizing the localization process. The recovery of the density matrix in approximately diagonal form enables us to verify the approach to thermal equilibrium in the long time limit and we comment on the connection between QSD and the decoherent histories approach.

## 1 Introduction

One of the approaches to quantum theory, developed the last years, is the quantum state diffusion (QSD) picture [1, 2, 3]. The construction of this picture was motivated by stochastic reduction theories and the necessity of describing individual experimental outcomes. Mathematically it is equivalent to the Lindblad master equation (hence it applies only to the Markovian regime of open quantum systems). Its main feature is that it describes the stochastic evolution of an individual system in Hilbert space, a treatment complementary

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to the deterministic evolution provided by the density matrix. Being a phenomenological picture, it has proved within its domain of applicability to be in very good agreement with experiments involving individual systems [4] and it is helping us to obtain more physical intuition about processes of the microworld.

Whenever the master equation for the reduced dencity matrix $\hat{\rho}$ describing the evolution of the open system is of Lindblad form [5] there is a unique and consistent way of unravelling it into an ensemble of individual system state vectors each of which obeys a stochastic differential equation. Thus if the evolution of $\hat{\rho}$ is given by,

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \sum_n (\hat{L}_n\hat{\rho}\hat{L}_n^\dagger - \frac{1}{2}\hat{L}_n^\dagger\hat{L}_n\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}_n^\dagger\hat{L}_n) \tag{1}$$

the evolution for the state vector $|\psi\rangle$ of the individual system is given by the non-linear stochastic Itô differential equation,

$$|d\psi\rangle = \frac{i}{\hbar}\hat{H}|\psi\rangle dt + \sum_n (\langle\hat{L}_n^\dagger\rangle_\psi\hat{L}_n - \frac{1}{2}\hat{L}_n^\dagger\hat{L}_n - \frac{1}{2}\langle\hat{L}_n^\dagger\rangle_\psi\langle\hat{L}_n\rangle_\psi|\psi\rangle dt + \sum_n (\hat{L}_n - \langle\hat{L}_n\rangle_\psi)|\psi\rangle d\xi_n \tag{2}$$

Here $\hat{H}$ is the Hamiltonian of the system in the absence of environment, sometimes modified by terms depending on the Lindblad operators $\hat{L}_n$, $\hat{L}_n^\dagger$ which model the effect of the environment. The independent stochastic differentials $d\xi_n$ are complex and satisfy,

$$Md\xi_n = 0, \quad M(d\xi_n d\xi_m) = 0, \quad M(d\xi_n d\xi_m^*) = \delta_{mn}dt \tag{3}$$

In the above $\langle\rangle_\psi$ denotes mean value with respect to $|\psi\rangle$ and M expresses the average over the ensemble. The density matrix is recovered by the formula

$$\hat{\rho} = M(|\psi\rangle\langle\psi|) \tag{4}$$

and consistency between the two pictures means that when mean values taken with respect to $|\psi\rangle$ in QSD are averaged over the ensemble, results agree with those obtained using $\hat{\rho}$.

# 2 The Model

In this paper we apply the QSD picture to the class of models consisting of an harmonic oscillator in a finite temperature dissipative environment. The Hamiltonian of the system in the absence of the environment is given by,

$$H = \hbar\omega(\hat{\alpha}^\dagger\hat{\alpha} + \frac{1}{2}) \tag{5}$$

while time evolution is described by a Lindblad master equation (1) with two environment operators,

$$\hat{L}_1 = [(\overline{n}+1)\gamma]^{1/2}\hat{\alpha}, \qquad \hat{L}_2 = (\overline{n}\gamma)^{1/2}\hat{\alpha}^\dagger \tag{6}$$

Here $\hat{\alpha}$ and $\hat{\alpha}^\dagger$ are the annihilation and creation operators for the harmonic oscillator,

$$\hat{\alpha} = (\sigma_p/\hbar)\hat{q} + i(\sigma_q/\hbar)\hat{p} \tag{7}$$

where $\sigma_q$ and $\sigma_p$ denote the dispersions of position and momentum for a standard coherent state,

$$\sigma_q = (\frac{\hbar}{2m\omega})^{1/2}, \qquad \sigma_p = (\frac{\hbar m\omega}{2})^{1/2} \tag{8}$$

and

$$\overline{n} = [\exp(\hbar\omega/kT) - 1]^{-1} \tag{9}$$

while the coefficient $\gamma$ represents dissipation. The process under study could be the damping of a mode of the electromagnetic field in a cavity, coupled to a beam of two-level atoms, some of which are initially excited or the damping of a coherent light beam propagating in a weakly absorbing medium [6].

The treatment will be similar to that of reference [7]. There, the QSD picture was applied to quantum Brownian motion (QBM) models resulting in approximate stationary solutions to Itô equation. These were correlated coherent states. They minimize a more general uncertainty relation [8] and are localized in phase space. For quadratic potentials the solutions are exact and stability is global *i.e.* every initial state evolving under Itô takes the form of the stationary one after some time. One of the main results of this work was that, after localization time, every initial density matrix tends to the form,

$$\hat{\rho} = \int f(p,q,t)|\psi_{pq}\rangle\langle\psi_{pq}| \tag{10}$$

The above expression gives a natural way for approximately diagonalizing $\hat{\rho}$ in the same basis at all moments of time, expressing phase space localization in the density matrix language. It resembles the well known P-representation of the density matrix [9] but differs in that $|\psi_{pq}\rangle$ are correlated rather than standard coherent states and $f(p,q,t)$ is always positive by construction. The expected approach to thermalequilibrium was also proved in Ref. [7] and finally localization rates were studied and the connection with decoherent histories was exemplified using phase space projectors constructed from $|\psi_{pq}\rangle$.

The linearity of Lindblad operators in position $\hat{q}$ and momentum $\hat{p}$ makes our analysis resemble the one of QBM model but the physics is obviously not the same. Here we are studying a process, Markovian at any finite temperature T (even small) and not only in the Fokker-Plank limit (which is a high temperature limit), studied in ref. [7]. The effect of the environment here is incorporated into two Lindblad operators expressing different processes and the sources of phase space localization together with the localization rates turn out to be of a different nature in both cases.

The same system has also been subject of study previously [10, 11, 12]. In [12] the aproach to thermal equilibrium was shown numerically, while in [10] it was argued that we have coherent states as stationary solutions to Itô equation and a localization theorem was proved in [11]. Our study concentrates on diffferent aspects of the problem (and we also believe that the authors of [11] have made a mistake in their calculation). The purpose

of this paper is to study connections between localization and decoherence as in ref. [7] but in a model different to that one, and also to give an analytic account of the numerical results of ref. [12].

# 3 Stationary Solutions

Following [7] in the search for the stationary solution we require that they satisfy the condition,

$$|\psi\rangle + |d\psi\rangle = \exp[\frac{i}{\hbar}\hat{q}d\overline{p} - \frac{i}{\hbar}\hat{p}d\overline{q} + \frac{i}{\hbar}d\phi]|\psi\rangle \tag{11}$$

with,

$$d\phi = \phi_t dt + \sum_{n=1}^{2}(\phi_n d\xi_n + \phi_n^* d\xi_n^*) \tag{12}$$

$$d\overline{p} = -mw^2\ \overline{q}dt - (\gamma/2)\overline{p} + \sum_{n=1}^{2}[\sigma(\hat{p},\hat{L}_n)d\xi_n + \sigma(\hat{L}_n,\hat{p})d\xi_n^*] \tag{13}$$

$$d\overline{q} = \frac{\overline{p}}{m}dt - (\gamma/2)\overline{q} + \sum_{n=1}^{2}[\sigma(\hat{q},\hat{L}_n)d\xi_n + \sigma(\hat{L}_n,\hat{q})d\xi_n^*] \tag{14}$$

Condition (10) means that the shape of the stationary solution is preserved and only $\overline{p}$ and $\overline{q}$ change. Here $\phi$ is a phase and by $\overline{p}$ and $\overline{q}$ we mean $\langle\hat{p}\rangle_\psi$ and $\langle\hat{q}\rangle_\psi$ respectivelly. We have also introduced:

$$\sigma(\hat{\Gamma},\hat{O}) = \langle\hat{\Gamma}^\dagger\hat{O}\rangle - \langle\hat{\Gamma}^\dagger\rangle\langle\hat{O}\rangle \tag{15}$$

After lengthy but not difficult calculations condition (9) leads to,

$$\langle q|\overline{p}\ \ \overline{q}\rangle \equiv \psi(q) = \exp[-\Lambda(q-\overline{q})^2 + \frac{i}{\hbar}\overline{p}q] \tag{16}$$

with $\Lambda$ real,

$$\Lambda = \frac{1}{4\sigma_q^2} = \frac{m\omega}{2\hbar} \tag{17}$$

which is a standard (harmonic oscillator) coherent state characterized by minimum uncertainty:

$$\sigma_q^2\sigma_p^2 = \hbar^2/4 \tag{18}$$

This property makes the stationary solutions look like phase space points to classical eyes.

# 4 Localization Rates

The existence of stationary solutions gives rise to the questions of stability and localization rates. To prove that every state tends to the form of the stationary one, we observe that the stationary solution is eigenstate of $\hat{\alpha}$. To show global stability, we need to show that,

$$Md(\Delta\hat{\alpha})^2 \leq 0 \tag{19}$$

Now for a non Hermitian operator $\hat{O}$ we can define the spread as,

$$(\Delta\hat{O})^2 \equiv \sigma(\hat{O}, \hat{O}) \tag{20}$$

Using the two above equations we obtain,

$$Md(\Delta\hat{\alpha})^2 = \frac{\gamma}{2\hbar^2}(\overline{n}+1/2)[\hbar^2 - 4R^2 - 2\frac{\sigma_q^2}{\sigma_p^2}(\Delta\hat{p})^4 - 2\frac{\sigma_p^2}{\sigma_q^2}(\Delta\hat{q})^4] \tag{21}$$

where R, is the symmetrized correlation between $\hat{q}$ and $\hat{p}$,

$$R \equiv \frac{1}{2}\langle\{\hat{p}, \hat{q}\}\rangle - \langle\hat{p}\rangle\langle\hat{q}\rangle \tag{22}$$

while $(\Delta\hat{p})^2$ and $(\Delta\hat{q})^2$ denote the dispersions. Then setting,

$$(\Delta\hat{q})^2 = \sigma_q^2(1+Q) \qquad\qquad (\Delta\hat{p})^2 = \sigma_p^2(1+P) \tag{23}$$

clearly

$$P \geq -1, \qquad\qquad Q \geq -1 \tag{24}$$

and stationary solution is obtained for $P = Q = R = 0$. The dispersion of $\hat{\alpha}$ reads in terms of P and Q,

$$(\Delta\hat{\alpha})^2 = (P+Q)/4 \geq 0 \tag{25}$$

which substituted in (20) leads to

$$Md(\Delta\hat{\alpha})^2 = -2\gamma(\overline{n}+1/2)[\frac{R^2}{\hbar^2} + \frac{P^2}{8} + \frac{Q^2}{8} + (\Delta\hat{\alpha})^2] \tag{26}$$

The above expression is negative and vanishes only for a coherent state. This proves the global stability of the stationary solutions. Our expression for the rate of localization is not in agreement with the one obtained in [11].

It is obvious from (25) that the localization process is characterized by a timescale of order $t_l = [\gamma(\overline{n}+1/2)]^{-1}$. This coresponds to a minimum rate of localization,

$$M\frac{d(\Delta\hat{\alpha})^2}{dt} \leq -2\gamma(\overline{n}+1/2)(\Delta\hat{\alpha})^2 \tag{27}$$

Then using expression (8) we obtain,

$$t_l \sim \frac{1}{\gamma}\tanh(\hbar\omega/2kT) \tag{28}$$

which for $\hbar\omega << kT$ gives,

$$t_l \sim \frac{\hbar\omega}{\gamma kT} \tag{29}$$

while for $\hbar\omega >> kT$ we have,

$$t_l \sim \frac{1}{\gamma} \tag{30}$$

When studying the approach to thermal equilibrium the meaning of (30) will become evident. In the case of an initial state consisting of a superposition of wavepackets a large distance $d$ apart, $(\Delta\hat{q})^2 \sim d^2$ and term $Q^2$ will be the dominant one in (25). Then in the high temperature limit, we obtain by virtue of (22) and (24),

$$t_l \sim \frac{\hbar^2}{d^2 m\gamma kT} \tag{31}$$

This result agrees with the usual "decoherence time" for high $T$ [16]

# 5 Thermal Equilibrium

The approach to equilibrium in Ref. [12] was shown by computing numericaly the behaviour of $\langle n\rangle$ and the occupancy probabilities for various number states $|n\rangle$. Using our stationary solution we may do the same analitically.For times greater than the localization time, $\langle \hat{n(t)}\rangle_\psi$ is given by,

$$\langle \hat{n(t)}\rangle_\psi \equiv \langle \hat{n(t)}\rangle_{st} = \frac{\sigma_p^2}{\hbar^2}\,\overline{q}^2 + \frac{\sigma_q^2}{\hbar^2}\,\overline{p}^2 = \alpha^*(t)\alpha(t) \tag{32}$$

Here $\alpha$ denotes the eigenvalue of $\hat{\alpha}$. Using equations (13) and (14) we may derive an evolution equation for $\alpha(t)$ [13],

$$\alpha(t) = \alpha(0)\exp[-(i\omega+\gamma/2)t) + \sqrt{\overline{n}\gamma}\int_0^t \exp[-(i\omega+\gamma/2)(t-t')]d\xi_2(t') \tag{33}$$

It is clear that in the long time limit only the part of the solution with the integral over the stochastic process will survive. Then $\langle\hat{n}(t)\rangle$ reads,

$$\langle\hat{n}(t)\rangle = \overline{n}\gamma\int_0^t \exp[(\frac{\gamma}{2}+i\omega)\tau + (\frac{\gamma}{2}-i\omega)\tau']d\xi_2(\tau)d\xi_2^*(\tau'). \tag{34}$$

From the above expression only when we take the mean over the ensemble we are able to say that once equilibrium is reached $\langle\hat{n}\rangle$ will fluctuate around $\overline{n}$ as expected. Then the standard result,

$$M\langle\hat{n}(t)\rangle_{st} = \overline{n} \tag{35}$$

is recovered,since $M[d\xi_2(\tau)d\xi_2^*(\tau')] = \delta(\tau-\tau')d\tau d\tau'$. Using our stationary solution, in principle, we are able to compute time averages over the stochastic process and thus reproduce the numerical result of Ref. [12] for the occupation probabilities analytically. (This result is needed to exhibit the thermal nature of the fluctuations in (34)). However, the manipulations such a computation would involve are cumbersome. We shall then proceed

using the density operator to treat this problem. We may recover $\hat{\rho}$ following the treatment of Ref [7]. Once localization has taken place Eq$^{\mathrm{n}}$ (4) is of the form,

$$\hat{\rho} = \int f(p,q,t)|pq\rangle\langle pq| \tag{36}$$

Hence every initial $\hat{\rho}$ approaches the phase space diagonal form (36) on the localization timescale. Equation (36) is the P-representation of $\hat{\rho}$ with the nice property that $f(p,q,t)$ is positive. A Fokker-Plank equation is easily derived for $f(p,q,t)$ [13]:

$$\frac{\partial}{\partial t}f = -\frac{p}{m}\frac{\partial}{\partial q}f + \frac{\gamma}{2}\frac{\partial}{\partial q}(qf) + m\omega^2 q\frac{\partial}{\partial p}f + \frac{\gamma}{2}\frac{\partial}{\partial p}(pf) + \frac{\hbar\overline{n}\gamma}{2m\omega}\frac{\partial^2}{\partial q^2}f + \frac{\hbar}{2}\overline{n}\gamma m\omega\frac{\partial^2}{\partial p^2}f \tag{37}$$

It admits a unique Gaussian stationary solution being globally stable and approached for times t $>> \gamma^{-1}$

$$f_s(p,q) = \frac{1}{2\pi\overline{n}}exp(-\frac{1}{\overline{n}}|\alpha|^2) \tag{38}$$

This is the P-symbol expression for a thermal density operator. Hence every initial state approaches thermal $\hat{\rho}$ for long times, as expected. Then the elements $\langle n|\hat{\rho}|n\rangle$ may be evaluated and give the standard result,

$$\langle n|\hat{\rho}|n\rangle = \frac{1}{\overline{n}}(1+\overline{n}^{-1})^{-(n+1)} \tag{39}$$

for the occupation probabilities, since the probability of observing $n$ quanta in a coherent state(*i.e.* $|\langle n|pq\rangle|^2$) is given by the poissonian distribution with mean $|a|^2$. We note that our result together with that of Ref. [12] (which is numerical and for a specific initial state) give the first traces of a proof that the ergodic hypothesis holds for our system.

One of the featurs of our model is that it is applicable at low temperatures as well.When the temperature $T \rightarrow 0$, we see from equations (8), (33) and (36) that the system decays to its ground state as expected. (for zero temperature $f_s(p,q)$ becomes infinitely sharp around $\alpha = 0$ as seen from (38). A very interesting result obtained in connection with equation (30) is that localization and relaxation to thermal equilibrium proceed essentialy on the same time scale for $T \rightarrow 0$. This should be expected since for $\overline{n} = 0$, $\hat{L}_1 = \gamma\hat{\alpha}$ and $\hat{L}_2 = 0$, expressing only dissipation of energy to the environment. Therefore in this limit the mechanism of decoherence is not very efficient, exhibiting the deeper connection between environmentaly induced decoherence and noise.

# 6 Comments

In the context of explaining the emergence of classical behaviour as a result of the interaction of a system with its environment, the approaches of "decoherence of density operators" [14, 15, 16, 17] and of "decoherent histories" [18, 19, 20, 21, 22] give the frameworks for a systematic treatment. In the former the approximate diagonilization of the reduced density operator, expressed by (36) for this class of models is essential while in the latter the

decoherence of histories is the prerequisite to emergent classicality. The close connection between QSD and decoherent histories [23] can be illustrated using the results obtained so far. The localization properties of the stationary solutions to Itô equation makes natural the study of phase space histories. We thus take phase space projectors of the form,

$$\hat{P}_{pq} = \int_{\Gamma_{pq}} dpdq \ \ |pq\rangle\langle pq| \tag{40}$$

Here $\Gamma_{pq}$ is a phase space cell. Obviously those projectors are only approximate ones, the validity of approximation depending on the area covered by the cell and the nature of its boundary [20]. Construct now histories with the above projectors and assume our initial state is $\hat{\rho}$, then the decoherence functional is given by,

$$D(\underline{a}, \underline{a'}) = Tr(P_{a_n} K_{t_{n-1}}^{t_n} [P_{a_{n-1}} .... K_{t_1}^{t_2} [P_{a_1} K_{t_0}^{t_1} [\rho_0] P_{a'_1}] .... P_{a'_{n-1}}] P_{a'_n}) \tag{41}$$

Here $\underline{a}, \underline{a'}$ denote the two strings of projections at succesive moments of time $t_1....t_n$, $a_k$ characterizes the phase space cell at a time moment $t_k$ and $K_{t_{k-1}}^{t_k}$ is the propagator of the density operator obeying the master equation of our model. $D(\underline{a}, \underline{a'})$ given by (42) is recovered from the decoherence functional of the closed system after having traced out the environment and under the assumptions that the initial density operator factorizes and the process is Markovian.

For intervals between projections longer than the localization time the density operator will always evolve to the diagonal form (37). Studying then (42) we conclude that because of the diagonality of $\hat{\rho}$ the off- diagonal elements $D(\underline{a}, \underline{a'})$ will be approximatelly equal to zero. Therefore approximate decoherence is achieved. Then probabilities computed are found to be most strongly peaked around the classical path and are essentialy the same for both approaches provided we always refer to scales grater than $\frac{\hbar}{2}$. This is because the probabilities of phase space histories (computed from the diagonal elements of $D$) and the probabilities in QSD are both equivalent to the probabilities derived from the description of our system using the Fokker-Planck equation (38). Thus all the conclusions of the analysis of Ref. [7] are valid here as well. However, a comment we would like to make is that in our model localization and decoherence occur in the Lindblad operator while in Ref. [7] do not. This is due to the fact that here the eigenstates of $\hat{L}$ are preserved by the Hamiltonian while in general this need not be the case.

In connection with the results obtained in [7] we can then conclude that the diagonality of $\hat{\rho}$ in a basis consisting of eigenstates of the same operator at all momements of time, the construction of decohering phase space histories (requirments for emergent classicality) and the study of the approach to thermal equilibrium may be acomplished in a natural way using the stationary solutions to Itô equation. Hence the description of processes in terms of individual systems, the localization properties of its solutions and the close connection with decoherent histories, make quantum state diffusion very useful both on intuitive and calculational grounds in the study of the emergence of classical behaviour of open quantum systems.

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